# Circular Error in a Pendulum

What _causes_ circular error? A search on Google provided many links to articles on the subject, but I could not find any that said "why." This page is designed to answer that question in a reader-friendly manner: if you want to, you can scroll down when you get to the maths, skip the maths, go to the graphs and continue reading. This page is also designed for readers with smartphones, which is why some of the images are presented smaller than original.

There are numerous forces acting upon a pendulum as it swings back and forth in a mechanical clock, such as gravity, elastic energy from the suspension spring, and air resistance. On this page, the subject is gravity. It is well known among clockmakers that when the angle of swing changes, the accuracy of the timekeeping changes in clocks. The effect of gravity upon timekeeping accuracy can be measured by finding how long it takes for the pendulum to go over and back. The time it takes to go over and back is known as the period.
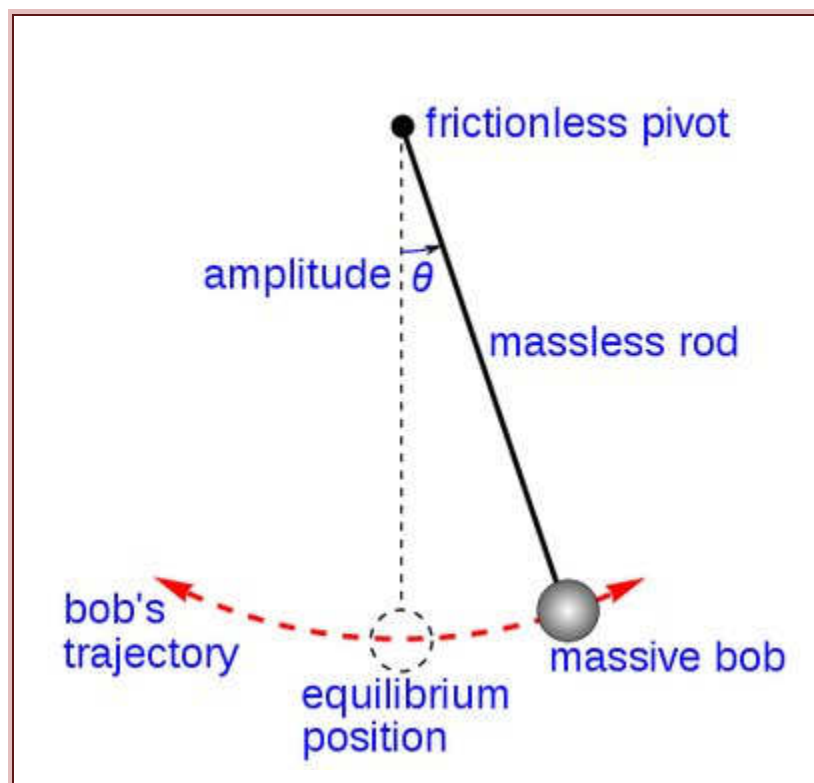
The moment a pendulum is released, gravity pulls it straight down, and a portion of that pull acts to pull the pendulum towards its vertical position, depending on the angle, the length of the pendulum, and the force of gravity, given by the following equation, in which acceleration is theta with an umlaut (two dots) over it:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

In a computer spreadsheet, the acceleration acting upon the pendulum can therefore be calculated for each moment in time, a small fraction of a second. Then the speed (angular velocity) of the pendulum is found by multiplying the acceleration by the amount of time (a small fraction of a second). The distance traveled (arc) in that amount of time is found by multiplying the angular velocity by the amount of time. The angle in radians is found by dividing the distance by the length of the pendulum. Then the angle in degrees is found by multiplying the angle in radians by 180 and dividing by pi (3.14). The results come together in a chart like this one.

	time		2	period in seconds			
6	time		2	period in seconds			
7	0.0001	0.993621	metres length		9.80665	gravity	
8	time	acc	vel	angle	radians	d	sinΘ
9							
10	0	2.220178	0	13	0.226893	0.225446	0.224951
11	0.0001	2.220178	0.000222	13	0.226893	0.225446	0.224951
12	0.0002	2.220177	0.000444	13	0.226893	0.225445	0.224951
13	0.0003	2.220177	0.000666	12.99999	0.226893	0.225445	0.224951
14	0.0004	2.220176	0.000888	12.99999	0.226893	0.225445	0.224951
15	0.0005	2.220175	0.00111	12.99998	0.226892	0.225445	0.224951
16	0.0006	2.220173	0.001332	12.99997	0.226892	0.225445	0.224951
17	0.0007	2.220172	0.001554	12.99996	0.226892	0.225445	0.22495
18	0.0008	2.22017	0.001776	12.99995	0.226892	0.225445	0.22495
19	0.0009	2.220168	0.001998	12.99994	0.226892	0.225445	0.22495
20	0.001	2.220166	0.00222	12.99993	0.226892	0.225444	0.22495
9998	0.9988	-2.22016	0.002714	-12.9999	-0.22689	-0.22544	-0.22495
9999	0.9989	-2.22017	0.002492	-12.9999	-0.22689	-0.22544	-0.22495
10000	0.999	-2.22017	0.00227	-12.9999	-0.22689	-0.22544	-0.22495
10001	0.9991	-2.22017	0.002048	-13	-0.22689	-0.22544	-0.22495
10002	0.9992	-2.22017	0.001826	-13	-0.22689	-0.22544	-0.22495
10003	0.9993	-2.22017	0.001604	-13	-0.22689	-0.22545	-0.22495
10004	0.9994	-2.22017	0.001382	-13	-0.22689	-0.22545	-0.22495
10005	0.9995	-2.22018	0.00116	-13	-0.22689	-0.22545	-0.22495
10006	0.9996	-2.22018	0.000938	-13	-0.22689	-0.22545	-0.22495
10007	0.9997	-2.22018	0.000716	-13	-0.22689	-0.22545	-0.22495
10008	0.9998	-2.22018	0.000494	-13	-0.22689	-0.22545	-0.22495
10009	0.9999	-2.22018	0.000272	-13	-0.22689	-0.22545	-0.22495
10010	1	-2.22018	4.96E-05	-13	-0.22689	-0.22545	-0.22495
10011	1.0001	-2.22018	-0.00017	-13	-0.22689	-0.22545	-0.22495
10012	1.0002	-2.22018	-0.00039	-13	-0.22689	-0.22545	-0.22495
10013	1.0003	-2.22018	-0.00062	-13	-0.22689	-0.22545	-0.22495
10014	1.0004	-2.22018	-0.00084	-13	-0.22689	-0.22545	-0.22495
10015	1.0005	-2.22017	-0.00106	-13	-0.22689	-0.22545	-0.22495
10016	1.0006	-2.22017	-0.00128	-13	-0.22689	-0.22545	-0.22495
10017	1.0007	-2.22017	-0.0015	-13	-0.22689	-0.22544	-0.22495
10018	1.0008	-2.22017	-0.00173	-13	-0.22689	-0.22544	-0.22495
10019	1.0009	-2.22017	-0.00195	-12.9999	-0.22689	-0.22544	-0.22495
10020	1.001	-2.22017	-0.00217	-12.9999	-0.22689	-0.22544	-0.22495

The formula in each cell of the spreadsheet is written so that the angle can be changed and the rest of the chart will be re-calculated to find how long it takes for the pendulum to reach the other side. In this chart, the angle shown is 13° and it takes one second for the pendulum to reach the other side. That is expected because a one-second pendulum is 39.1 inches (0.994 metres) long. When the pendulum reaches the other side, it changes direction to swing back, and this is seen in the chart when the velocity becomes negative. You can click on the chart below to see it full size: make your own chart and experiment with it.

	A	B	C	D	E	F	G
1							
2							
3							
4							
5							
6	time	2	period in seconds				
7	0.0001	=E7*(((B6/(2*PI()))^2))	metres length		9.80665	gravity	
8	time	acceleration	velocity	angle	radians	distance	sinθ
9							
10	0	=($E$7/$B$7)*G10	0	13	=(D10*PI())/180	=B$7*E10	=SIN(E10)
11	=A10+A$7	=($E$7/$B$7)*G11	=C10+(B10*A$7)	=(E11*180)/PI()	=F11/B$7	=F10-(C11*A$7)	=SIN(E11)
12	=A11+A$7	=($E$7/$B$7)*G12	=C11+(B11*A$7)	=(E12*180)/PI()	=F12/B$7	=F11-(C12*A$7)	=SIN(E12)
13	=A12+A$7	=($E$7/$B$7)*G13	=C12+(B12*A$7)	=(E13*180)/PI()	=F13/B$7	=F12-(C13*A$7)	=SIN(E13)
14	=A13+A$7	=($E$7/$B$7)*G14	=C13+(B13*A$7)	=(E14*180)/PI()	=F14/B$7	=F13-(C14*A$7)	=SIN(E14)
15	=A14+A$7	=($E$7/$B$7)*G15	=C14+(B14*A$7)	=(E15*180)/PI()	=F15/B$7	=F14-(C15*A$7)	=SIN(E15)
16	=A15+A$7	=($E$7/$B$7)*G16	=C15+(B15*A$7)	=(E16*180)/PI()	=F16/B$7	=F15-(C16*A$7)	=SIN(E16)
17	=A16+A$7	=($E$7/$B$7)*G17	=C16+(B16*A$7)	=(E17*180)/PI()	=F17/B$7	=F16-(C17*A$7)	=SIN(E17)
18	=A17+A$7	=($E$7/$B$7)*G18	=C17+(B17*A$7)	=(E18*180)/PI()	=F18/B$7	=F17-(C18*A$7)	=SIN(E18)
19	=A18+A$7	=($E$7/$B$7)*G19	=C18+(B18*A$7)	=(E19*180)/PI()	=F19/B$7	=F18-(C19*A$7)	=SIN(E19)

When the angle is changed to 45°, it takes 1.0367 seconds for the pendulum to swing from one side to the other, almost 4% longer. Since time taken becomes longer, a clock would be seen to be losing time.

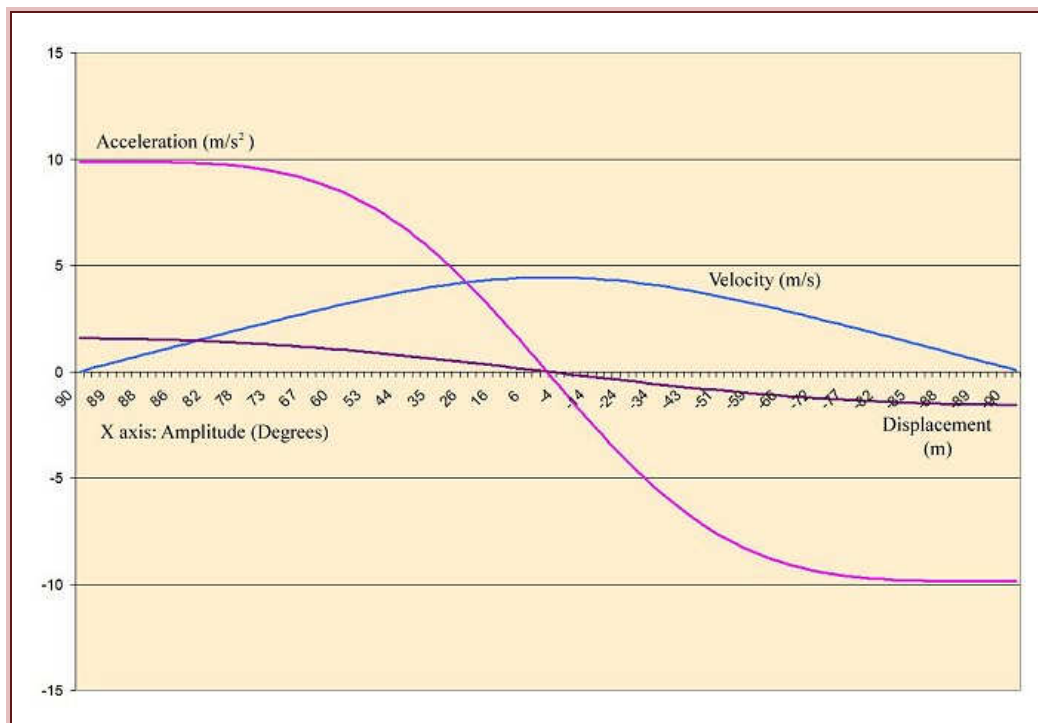| 6	time		2	period in seconds			
7	0.0001	0.993621	metres length		9.80665	gravity	
8	time	acc	vel	angle	radians	d	sinΘ
9							
10	0	6.978864	0	45	0.785398	0.780388	0.707107
11	0.0001	6.978864	0.000698	45	0.785398	0.780388	0.707107
12	0.0002	6.978863	0.001396	44.99999	0.785398	0.780388	0.707107
13	0.0003	6.978861	0.002094	44.99998	0.785398	0.780388	0.707106
14	0.0004	6.978859	0.002792	44.99996	0.785397	0.780388	0.707106
15	0.0005	6.978857	0.003489	44.99994	0.785397	0.780387	0.707106
16	0.0006	6.978854	0.004187	44.99992	0.785397	0.780387	0.707106
17	0.0007	6.97885	0.004885	44.99989	0.785396	0.780386	0.707105
18	0.0008	6.978847	0.005583	44.99986	0.785396	0.780386	0.707105
19	0.0009	6.978842	0.006281	44.99982	0.785395	0.780385	0.707105
20	0.001	6.978837	0.006979	44.99978	0.785394	0.780385	0.707104
10364	1.0354	-6.97883	0.008732	-44.9997	-0.78539	-0.78038	-0.7071
10365	1.0355	-6.97883	0.008034	-44.9998	-0.78539	-0.78038	-0.7071
10366	1.0356	-6.97884	0.007336	-44.9998	-0.78539	-0.78038	-0.7071
10367	1.0357	-6.97884	0.006639	-44.9998	-0.7854	-0.78039	-0.7071
10368	1.0358	-6.97885	0.005941	-44.9999	-0.7854	-0.78039	-0.70711
10369	1.0359	-6.97885	0.005243	-44.9999	-0.7854	-0.78039	-0.70711
10370	1.036	-6.97886	0.004545	-44.9999	-0.7854	-0.78039	-0.70711
10371	1.0361	-6.97886	0.003847	-44.9999	-0.7854	-0.78039	-0.70711
10372	1.0362	-6.97886	0.003149	-45	-0.7854	-0.78039	-0.70711
10373	1.0363	-6.97886	0.002451	-45	-0.7854	-0.78039	-0.70711
10374	1.0364	-6.97886	0.001753	-45	-0.7854	-0.78039	-0.70711
10375	1.0365	-6.97886	0.001055	-45	-0.7854	-0.78039	-0.70711
10376	1.0366	-6.97886	0.000358	-45	-0.7854	-0.78039	-0.70711
10377	1.0367	-6.97886	-0.00034	-45	-0.7854	-0.78039	-0.70711
10378	1.0368	-6.97886	-0.00104	-45	-0.7854	-0.78039	-0.70711
10379	1.0369	-6.97886	-0.00174	-45	-0.7854	-0.78039	-0.70711
10380	1.037	-6.97886	-0.00243	-45	-0.7854	-0.78039	-0.70711
10381	1.0371	-6.97886	-0.00313	-45	-0.7854	-0.78039	-0.70711
10382	1.0372	-6.97886	-0.00383	-44.9999	-0.7854	-0.78039	-0.70711
10383	1.0373	-6.97885	-0.00453	-44.9999	-0.7854	-0.78039	-0.70711

The position of the pendulum (its angle) affects how gravity acts upon it, and the result is that the acceleration changes continuously. You can click on each one of all the graphs below to see it full size.
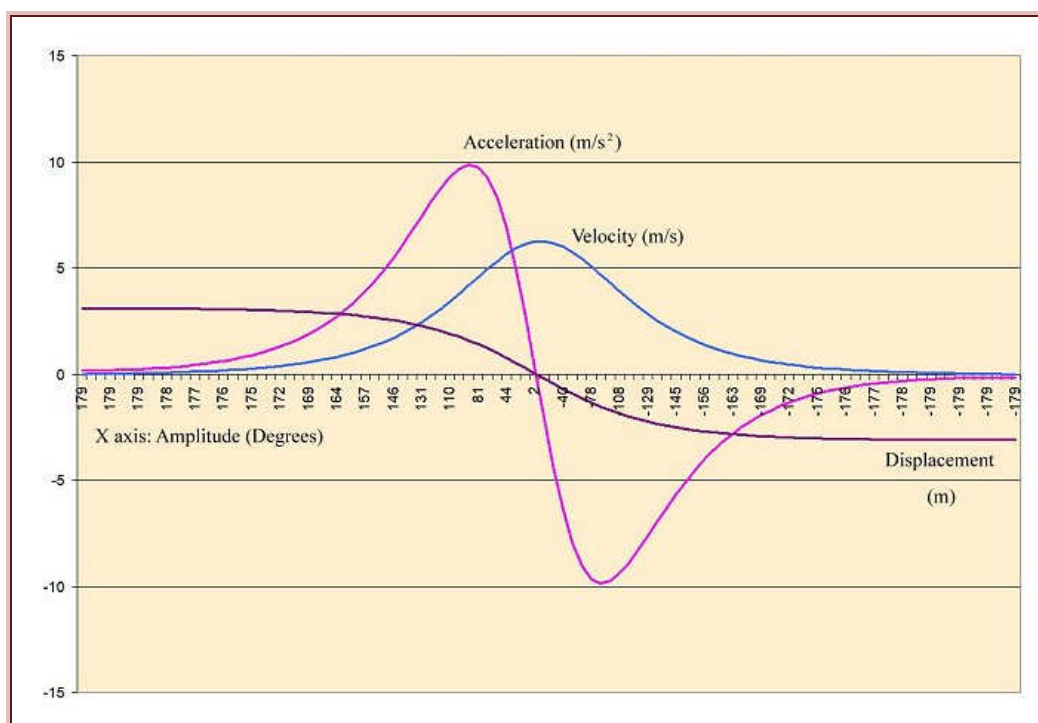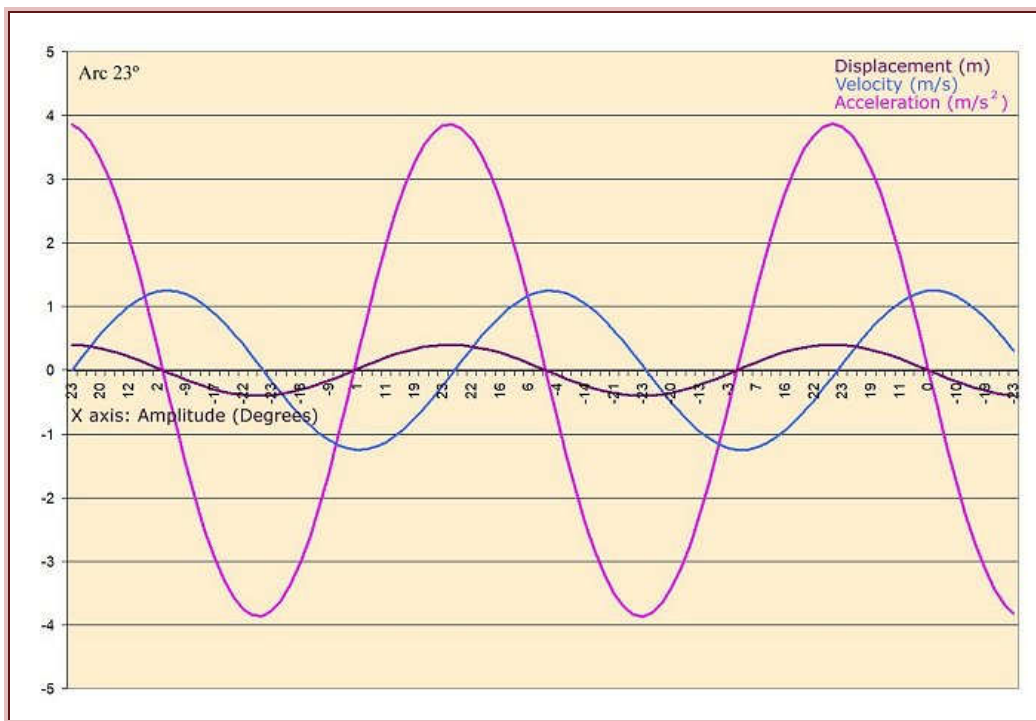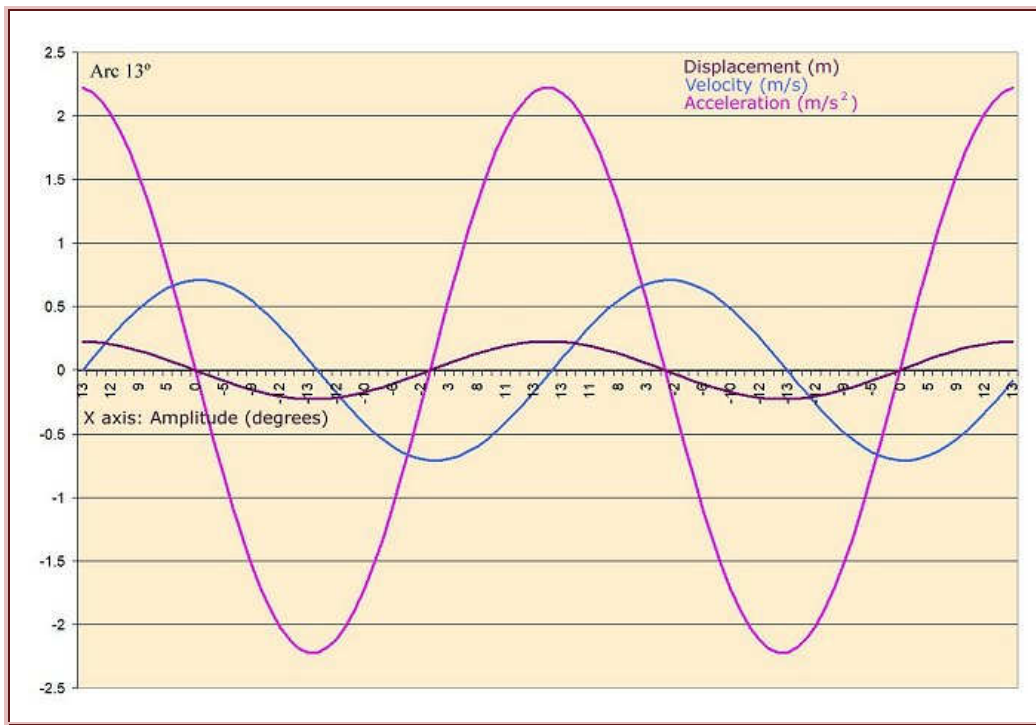
For a 13° arc from vertical:
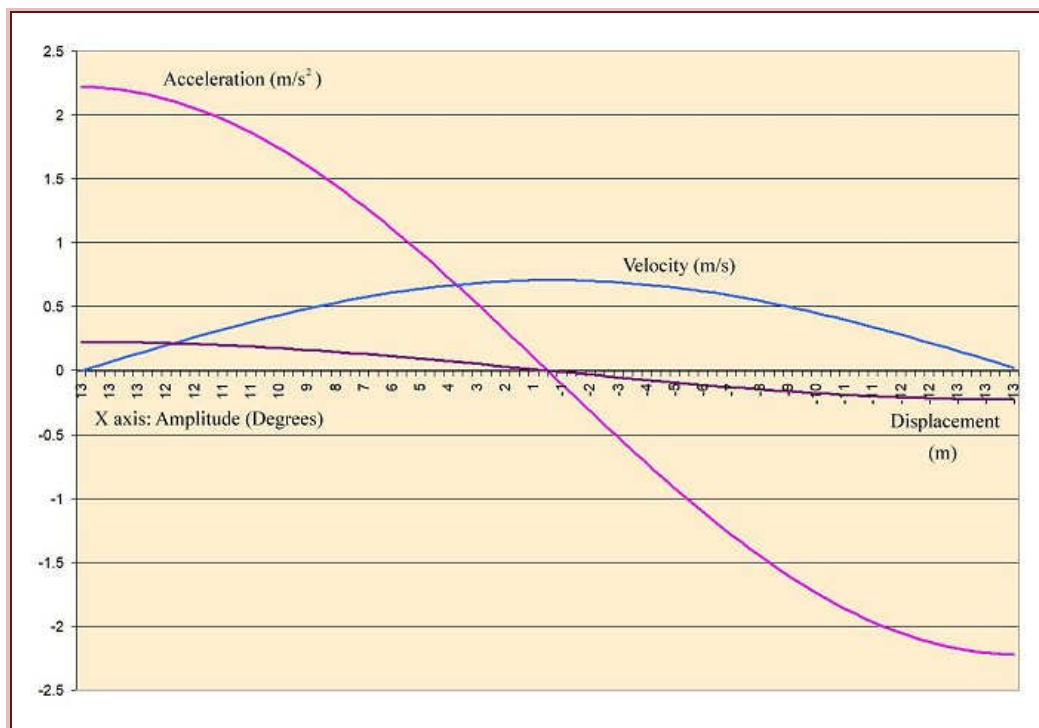
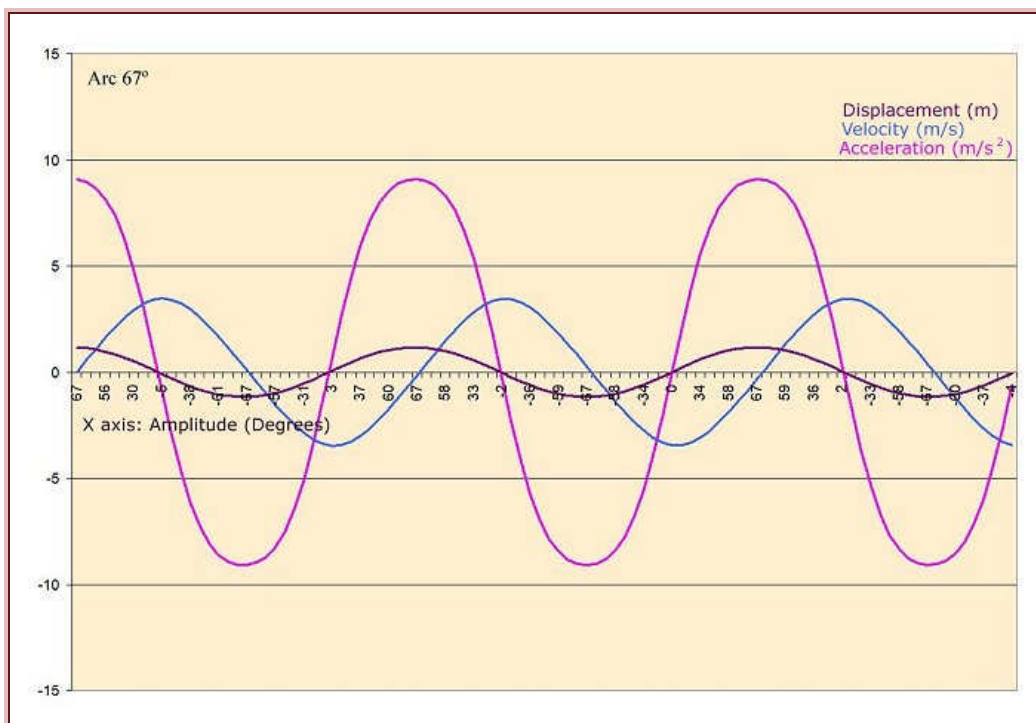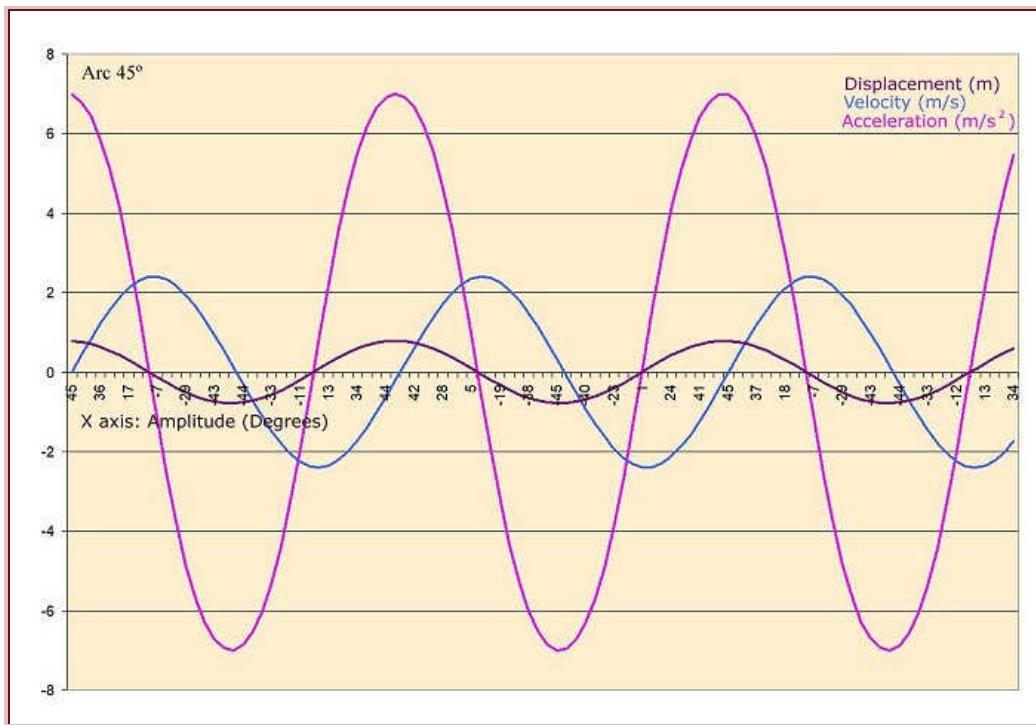The effect becomes more obvious for large angles.
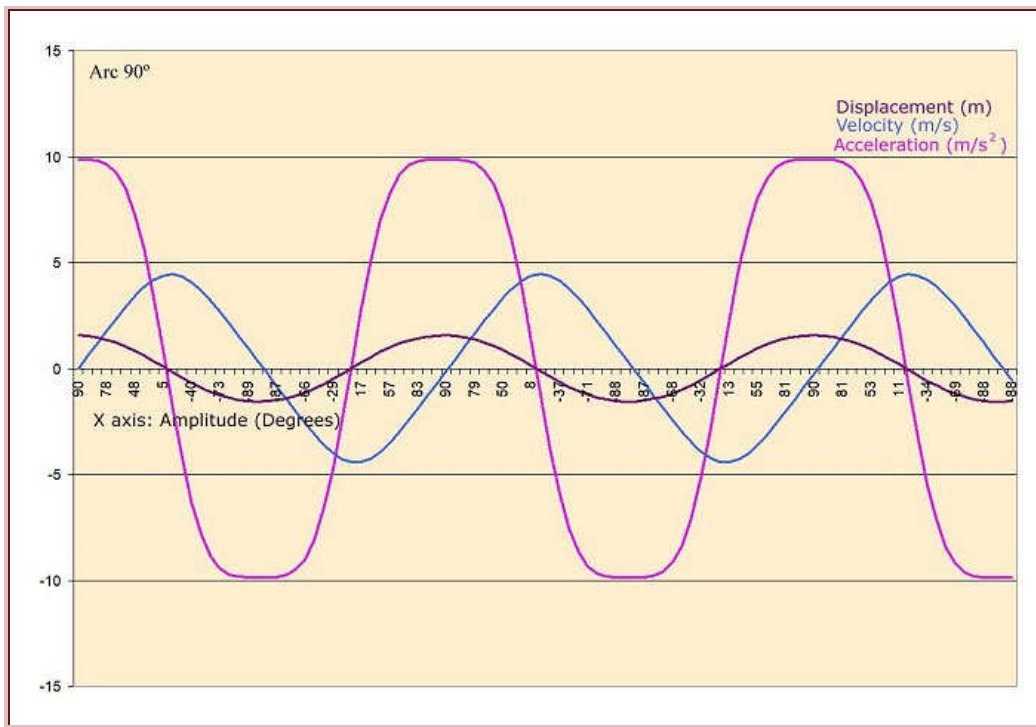
For a 90° arc from vertical:

## At 180°, the period goes to infinity.
For a 179° arc from vertical:

Looking at several graphs below, we can see that, as the angle increases, the height (amplitude) of the acceleration curve increases, but at a declining rate: when the angle doubles, the acceleration increases by less than double.

Beyond 90°, the acceleration curve no longer reaches a higher high, and forms what looks like a wave within a wave.
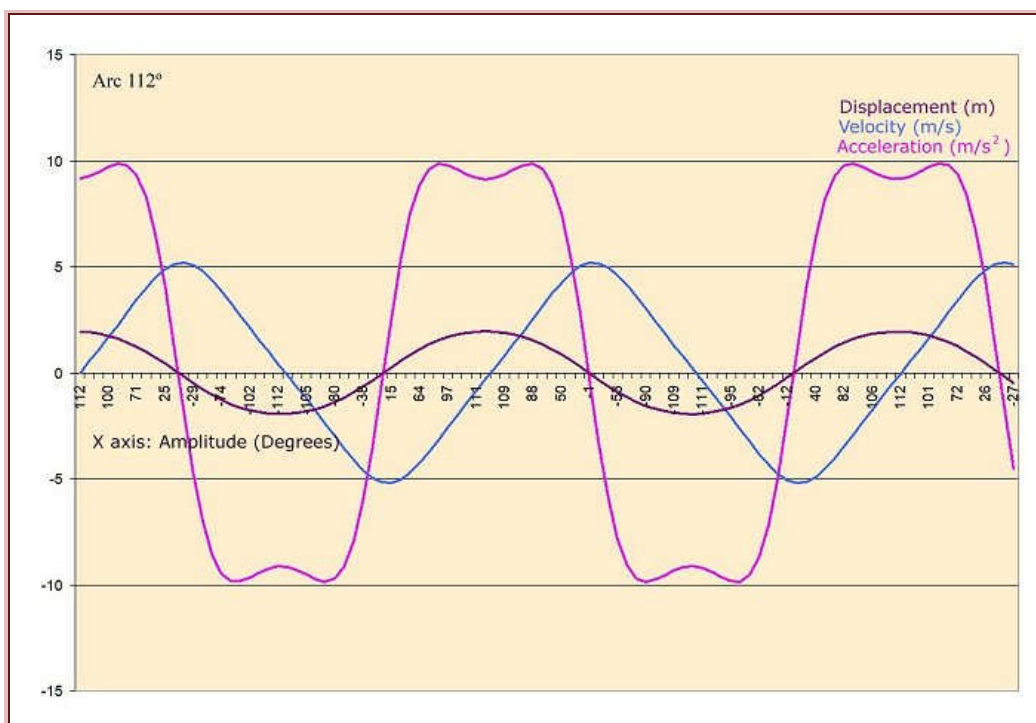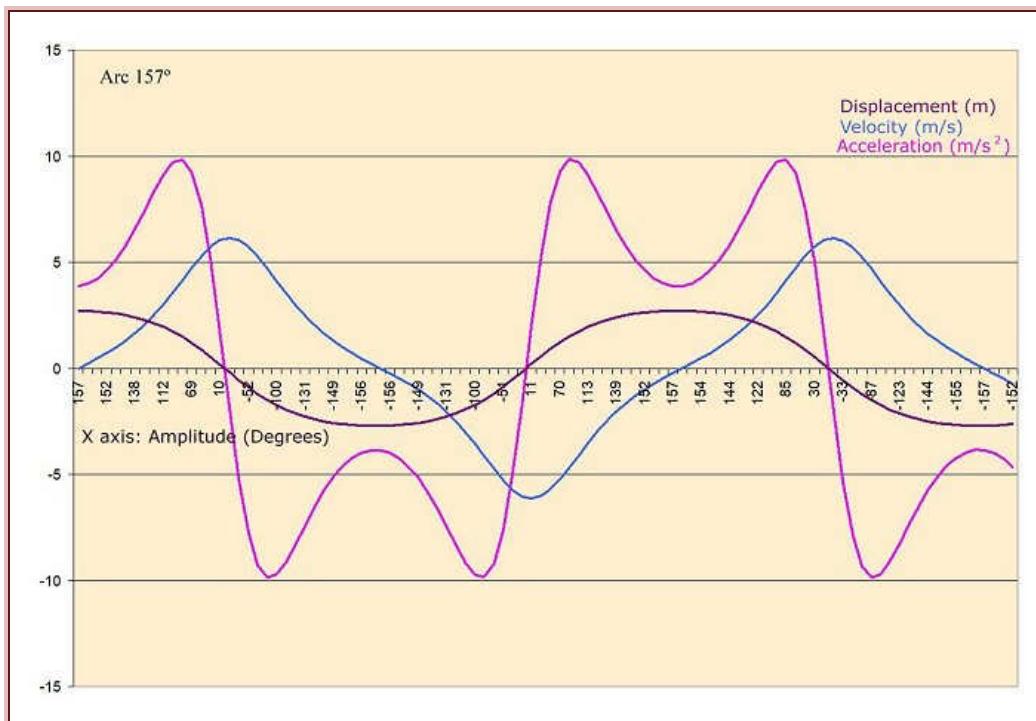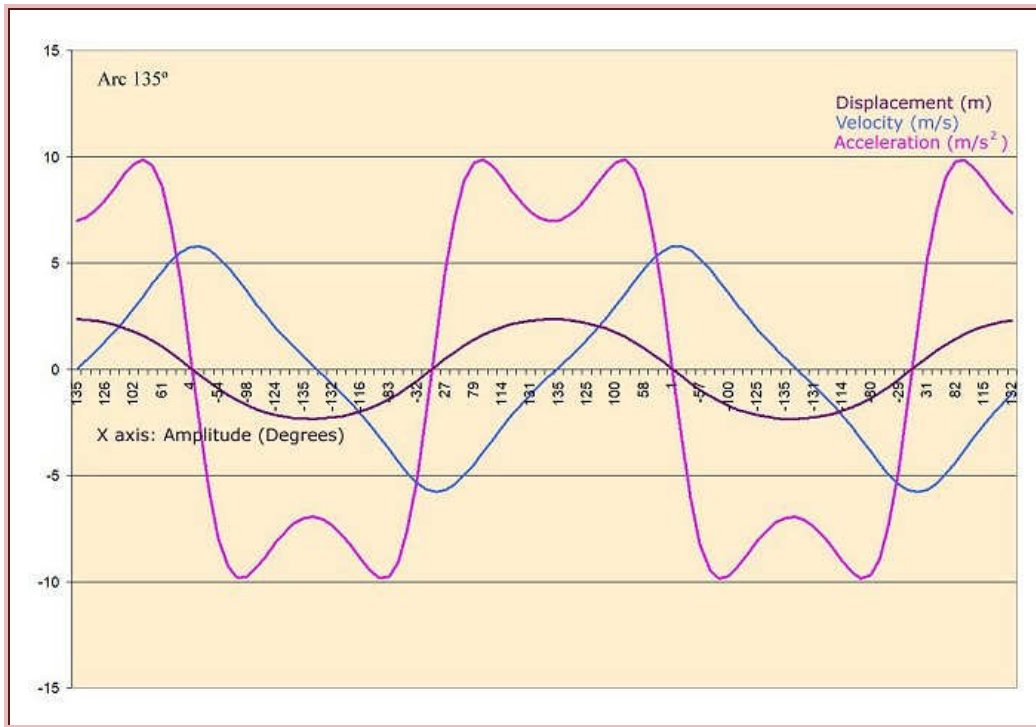
<p align="center">And around and around we go:</p>
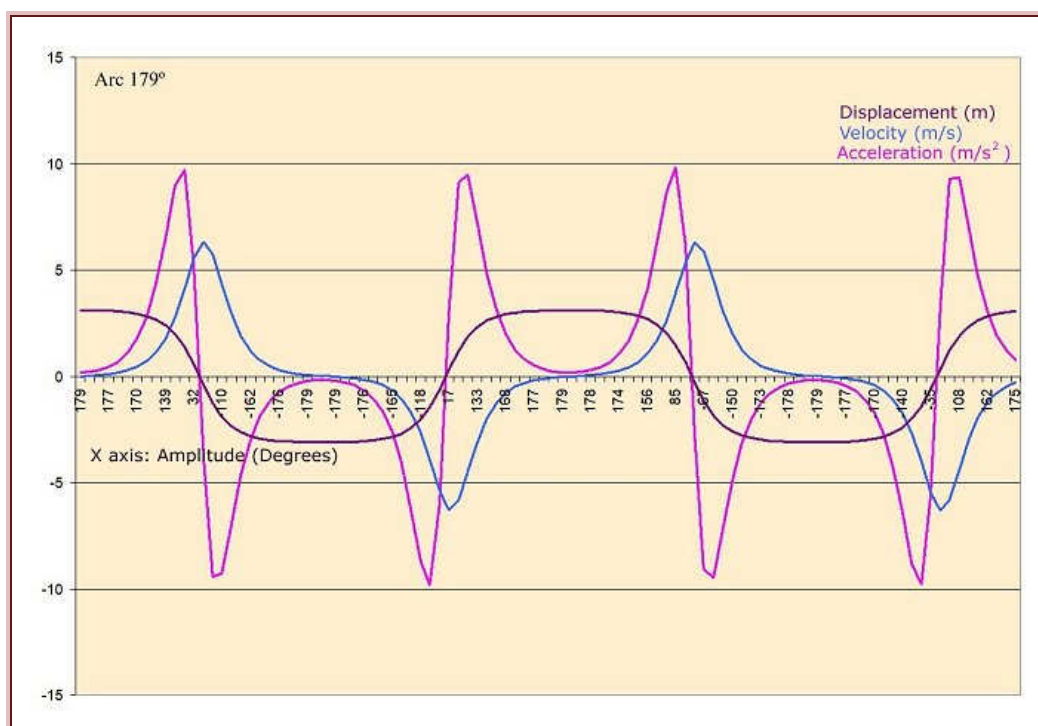
The Cause of Circular Error:

When the angle is doubled, the pendulum swings twice as far, but the acceleration does not double, and so the speed does not double either. If the pendulum swings twice as far and the speed does not double, then it takes <u>longer</u> to reach the other side: the period increases. The change in the period with changing angle is called circular error.

Note that I am referring to peak angular acceleration and peak angular velocity here.

The graphs below show how the period increases rapidly (exponentially) as the angle increases. The period is the time it takes to go back and forth, which is two seconds in this example. The curves are not perfectly smooth because of small rounding errors in the data.
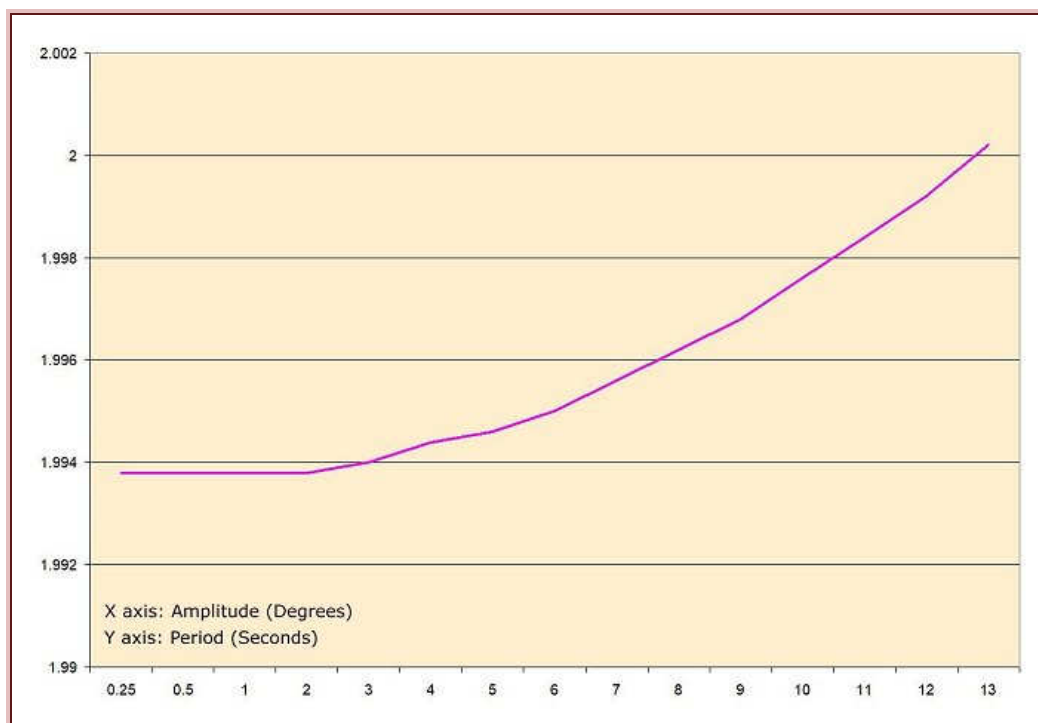
The graph shows that the period reaches two seconds when the angle reaches 13°. This means that the period is less than two seconds for smaller angles, causing a clock to gain time, but greater than two seconds for larger angles, causing a clock to lose time as a result of circular error. Notice that, for small angles, of 3° or less, the curve becomes flat because of what is called Linearization (when sinθ is approximately equal to θ).
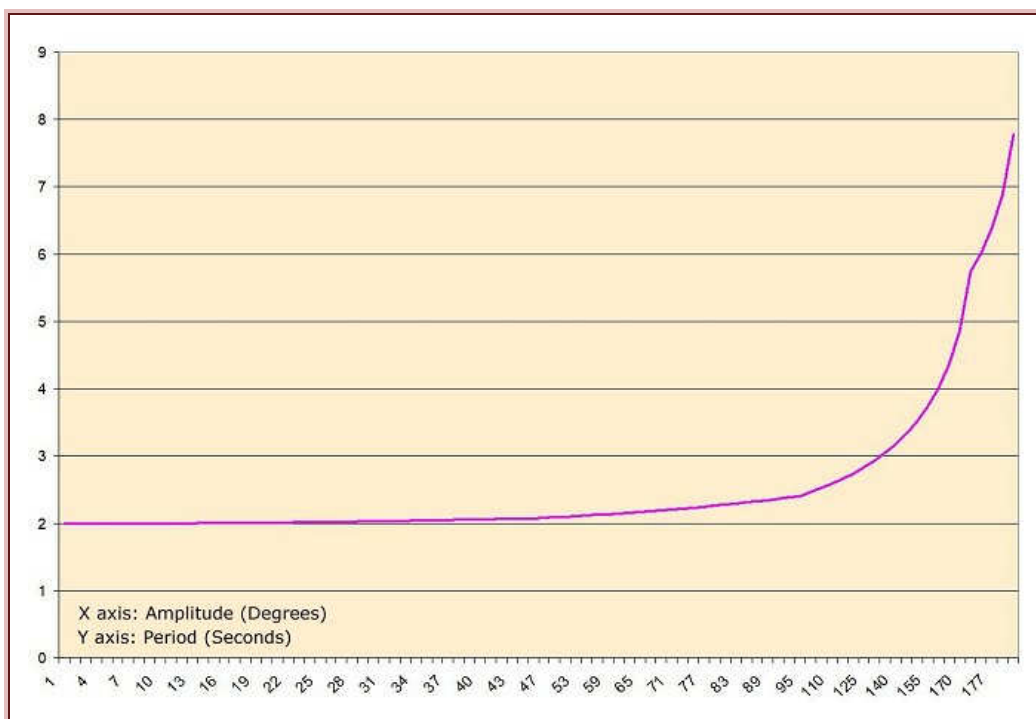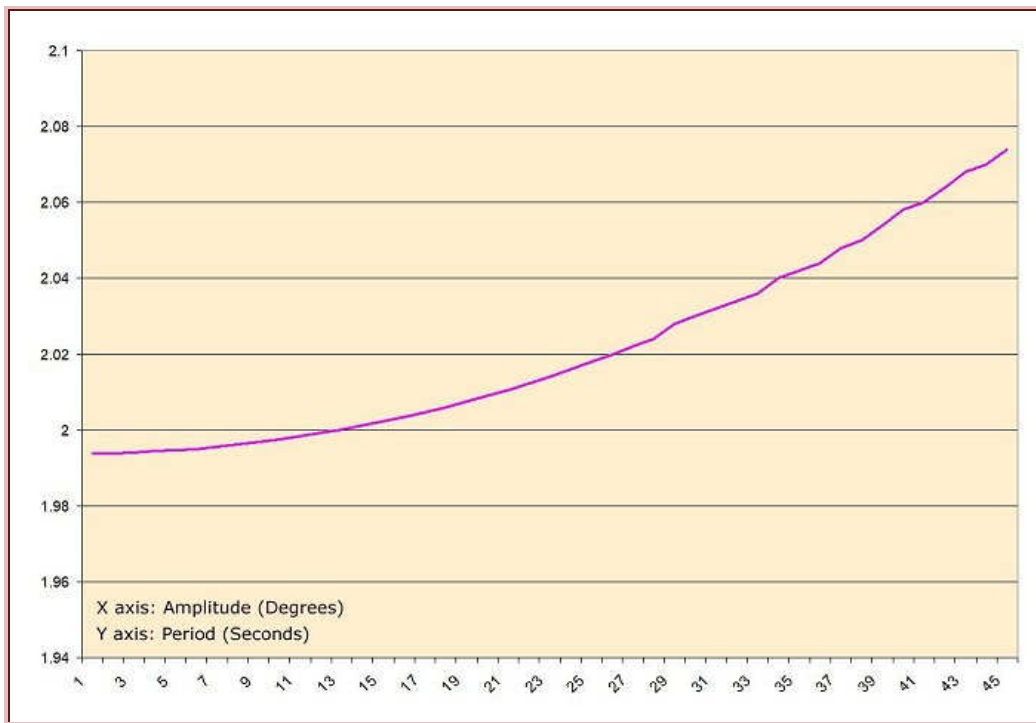
Changing the Length:

Performing similar calculations by changing the length of the pendulum, but keeping the amplitude unchanged at 13°, shows that the pendulum length

has a stronger effect upon circular error. In the chart below, the left column shows the expected period (time taken to go back and forth) in seconds. The next column shows the pendulum length in metres for the corresponding period. The right column shows the calculated period, using the spreadsheet at the top of this page.

period	length	time
0.25	0.015525	0.0314
0.5	0.062101	0.3752
1	0.248405	0.5002
2	0.993621	2.002
3	2.235648	4.5008
4	3.974486	8.0008

The graph below shows the expected period on the horizontal line (X axis) and the calculated period on the vertical line (Y axis). The data compares the one-second pendulum, which has an expected period and a calculated period of 2 seconds, with the periods for a pendulum of other lengths. The blue line shows what the line would look like without circular error (Y=X).
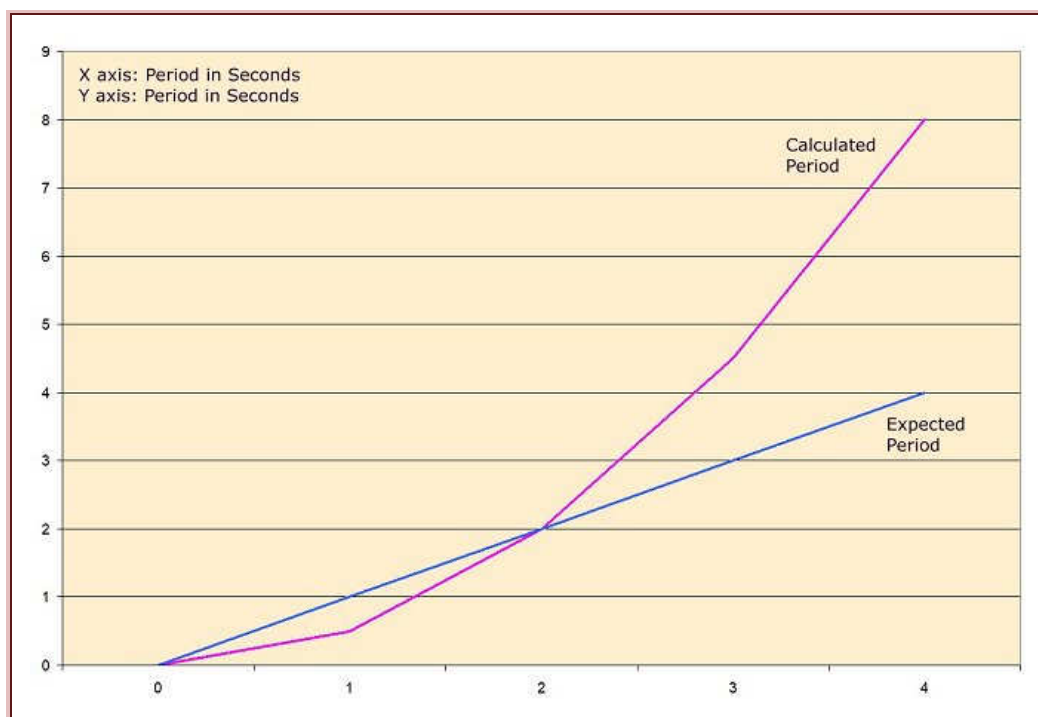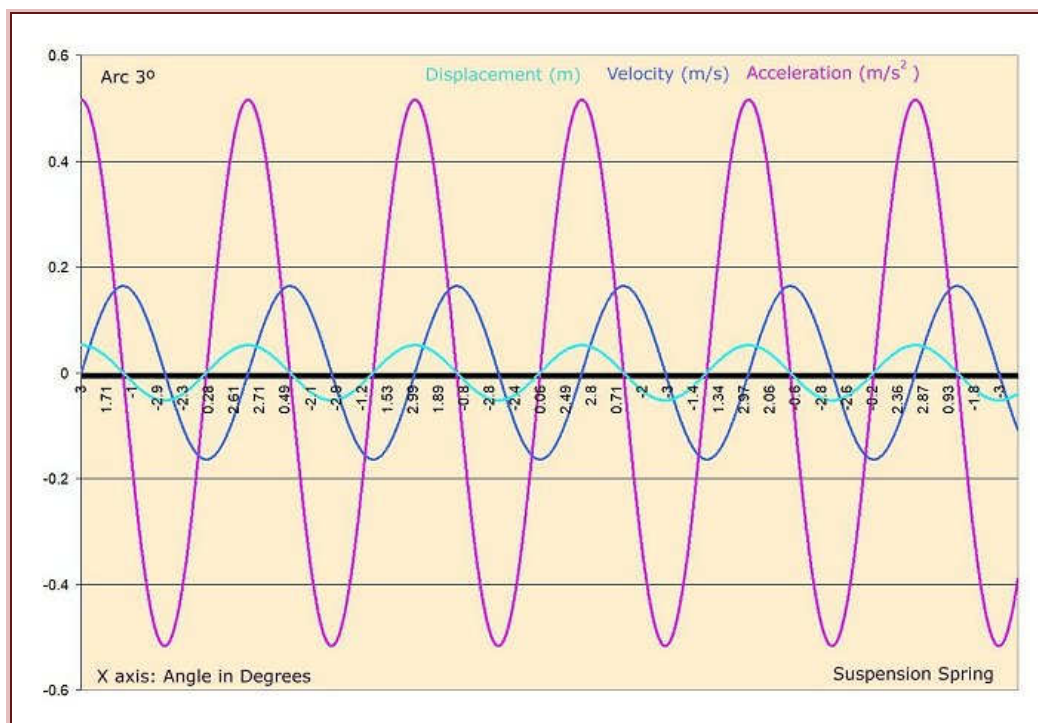
Adding a Suspension Spring:

Looking at the same one-second pendulum and adding a suspension spring, we get new graphs. The coefficient of elasticity of a spring deflected by bending remains constant, which is more or less true for small angles of 3° or less. In this example, I assume that the initial force applied by the suspension spring is equal to the initial force applied by gravity to bring the pendulum back to a vertical position, and the two forces are added together. Looking at a pendulum with an arc of only 3°, the graph for gravity looks like this one.
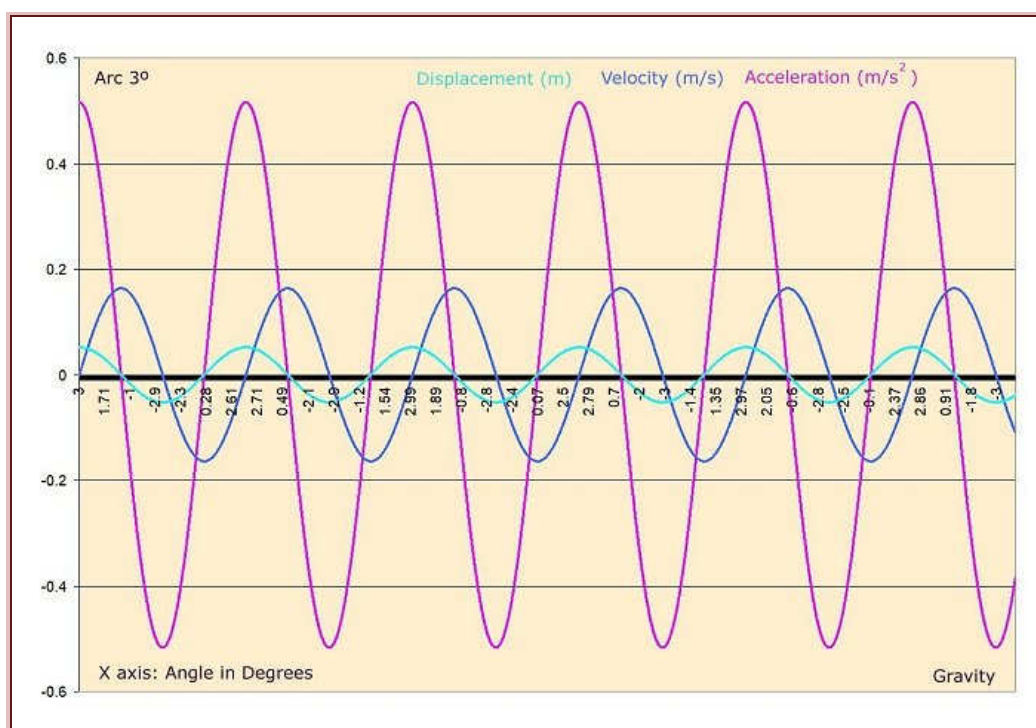
The next graph examines the action of a suspension spring alone (without gravity). This suspension spring exerts the same initial force as gravity in this example, and their initial accerations are therefore equal.

The graphs are identical and their acceleration waves overlap perfectly.

When the gravitational force and the force from the suspension spring are added together, peak acceleration is doubled because the two initial forces are equal in this example.
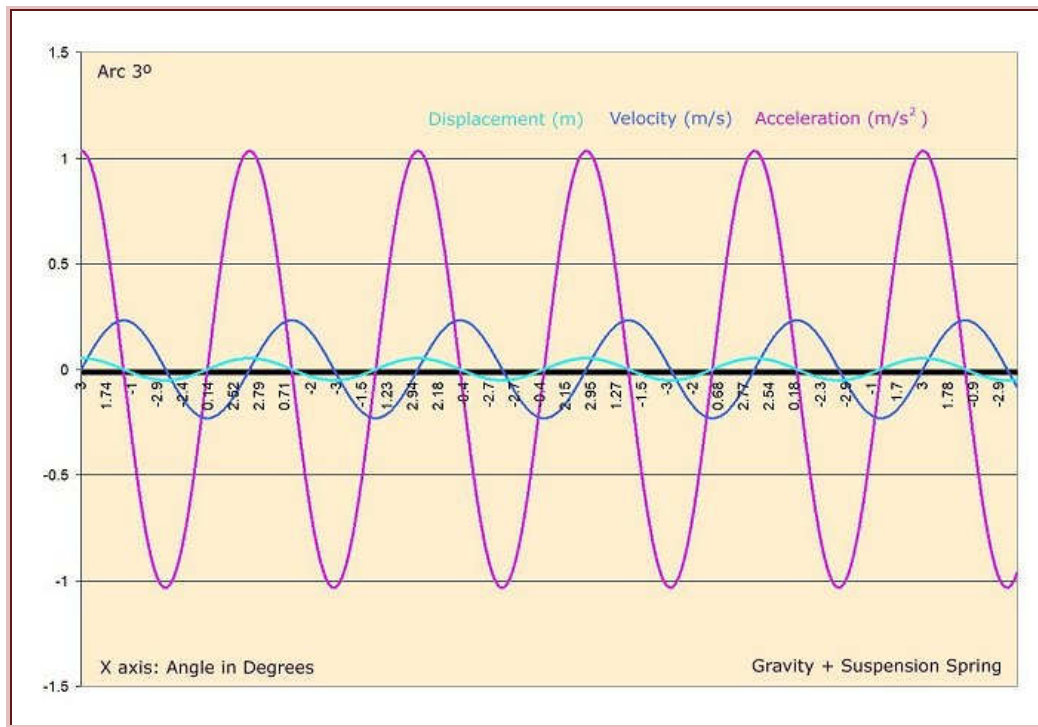
Peak velocity increases by 41%, but displacement remains unchanged.

	Gravity	Suspension Spring	Both
Acceleration	0.520	0.520	1.040
Velocity	0.164	0.164	0.232
Displacement	0.052	0.052	0.052

The period for the pendulum with the suspension spring added decreases from 2 seconds to 1.411 seconds. The pendulum length would need to be increased from 0.994m (39.1 inches) to 1.615m (63.6 inches) to get a period of 2 seconds with this suspension spring.

The data shows that, when the coefficient of elasticity of a spring deflected by bending remains constant, which is more or less true for small angles of 3° or less, then the effect of the suspension spring upon the period is constant, showing a horizontal line in the graph below. Since my 1992 Hermle 1161 grandfather clock has a pendulum amplutude of about 2.3° when powered by a 10.5 lb. weight, 1.8° when powered by a 5.5 lb. weight, and 1.1° when powered by a 3 lb weight, the angles are very small. Here is what a graph would look like if the coefficient of elasticity were constant for all angles.
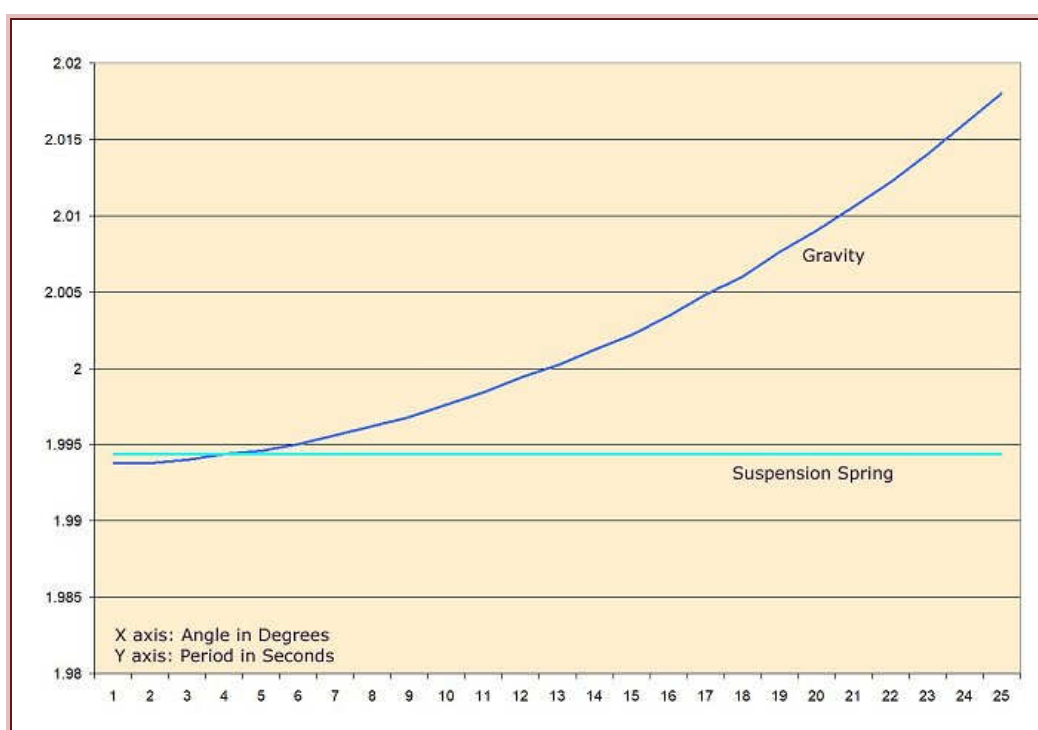
Adding a suspension spring reduces the period. If you choose a stronger suspension spring, it would reduce the period further.
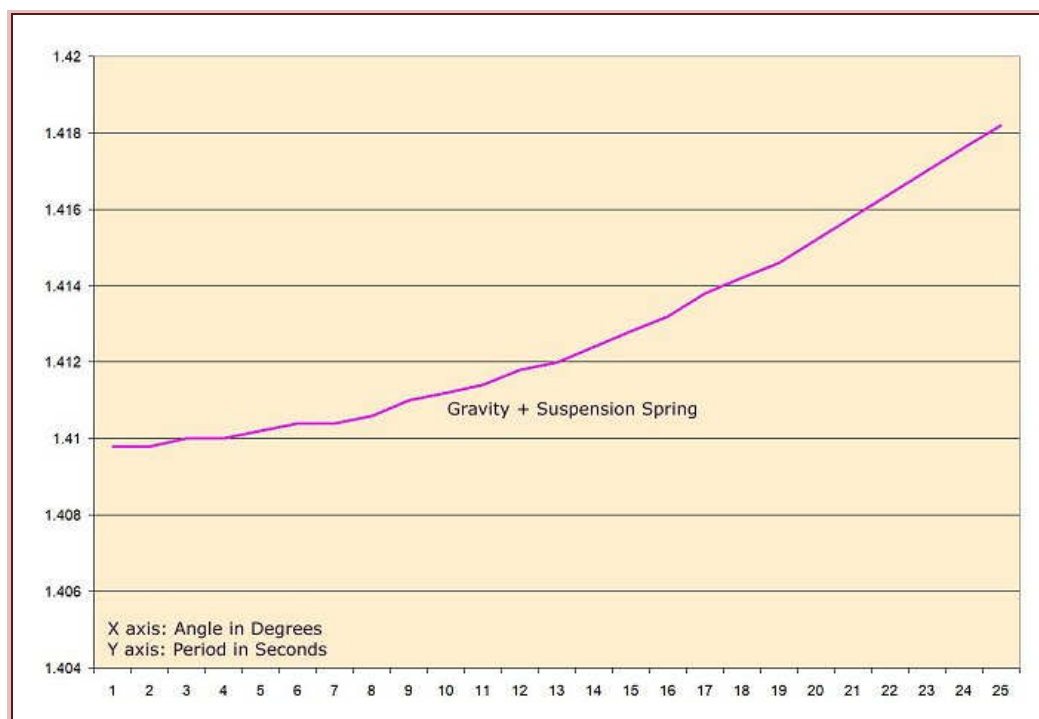
However, the coefficient of elasticity is NOT constant for larger angles. For example, if you double the angle from 45° to 90°, the force the suspension spring applies increases, but it does not double. The line for the suspension spring in the graph above should therefore curve upwards some as the angle increases.

In a graph for small angles, the difference between between a period of 1.4098 and a period of 1.4100 seconds is negligible.
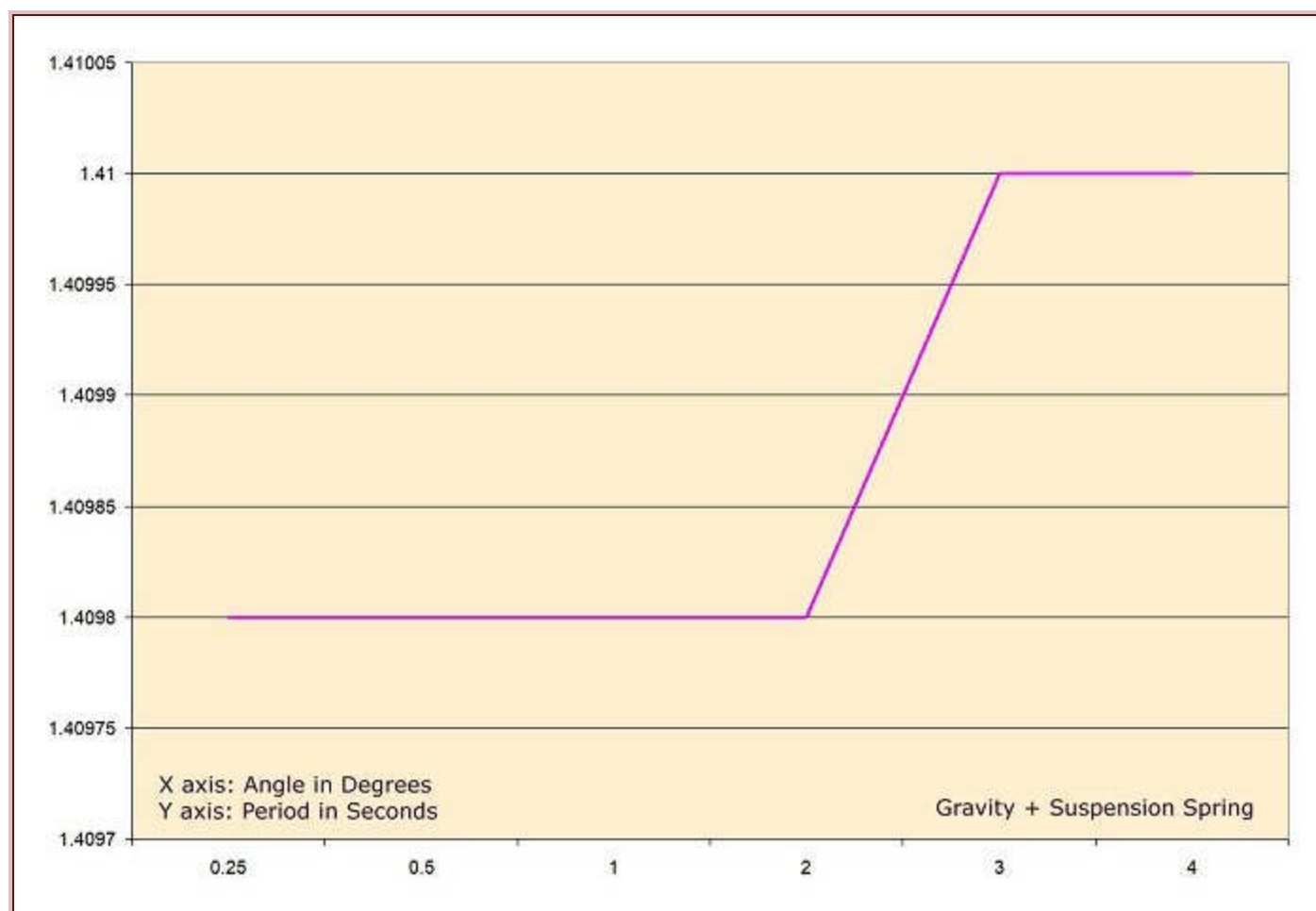
Adding an Escapement:

When an escapement is added to keep a pendulum going, as in a clock, the acceleration increases because there are now three forces acting upon the pendulum: the force of gravity, which is a function of $\sin\theta$, the elastic force from the suspension spring, which is a function of θ, and the force from the Graham escapement, as the escape wheel's tooth slides across the pallet's impulse face. In this example, the pallet has an arc of 6° and the pendulum

has an arc of 3°, so the peak acceleration from the escapement that reaches the pendulum is multiplied by cos2θ, which has a range of 0.997 and 1 when the arc is 6°. The acceleration from the escapement is therefore almost constant. When additional force is added, the peak angular velocity increases, so the period is reduced.

| Escapement Acceleration	Half Period
0	0.36
0.1	0.34
0.2	0.32
0.3	0.30
0.4	0.28

Furthermore, the behavior of the pendulum changes. When there is acceleration from gravity A_g and from the suspension spring A_s, acceleration falls to zero when the angle reaches zero. However, acceleration from the escapement A_e barely changes, so the velocity of the pendulum continues to increase until $A_e - A_g - A_s = 0$. Overall acceleration falls to zero when the forces cancel each other out. In one example, the total angle was 3° and all the forces cancelled each other out when the angle was 1.35°.

When a heavier weight is used to power a clock, the reduction in the period caused by the escapement is greater than the increase in the period caused by gravity, so the clock gains time.

For the same reason, an eight-day clock, powered by a mainspring and correctly adjusted, will gain time in the beginning of the week, and lose time towards the end of the week.

X axis: Time in Days
Y axis: Error in Seconds
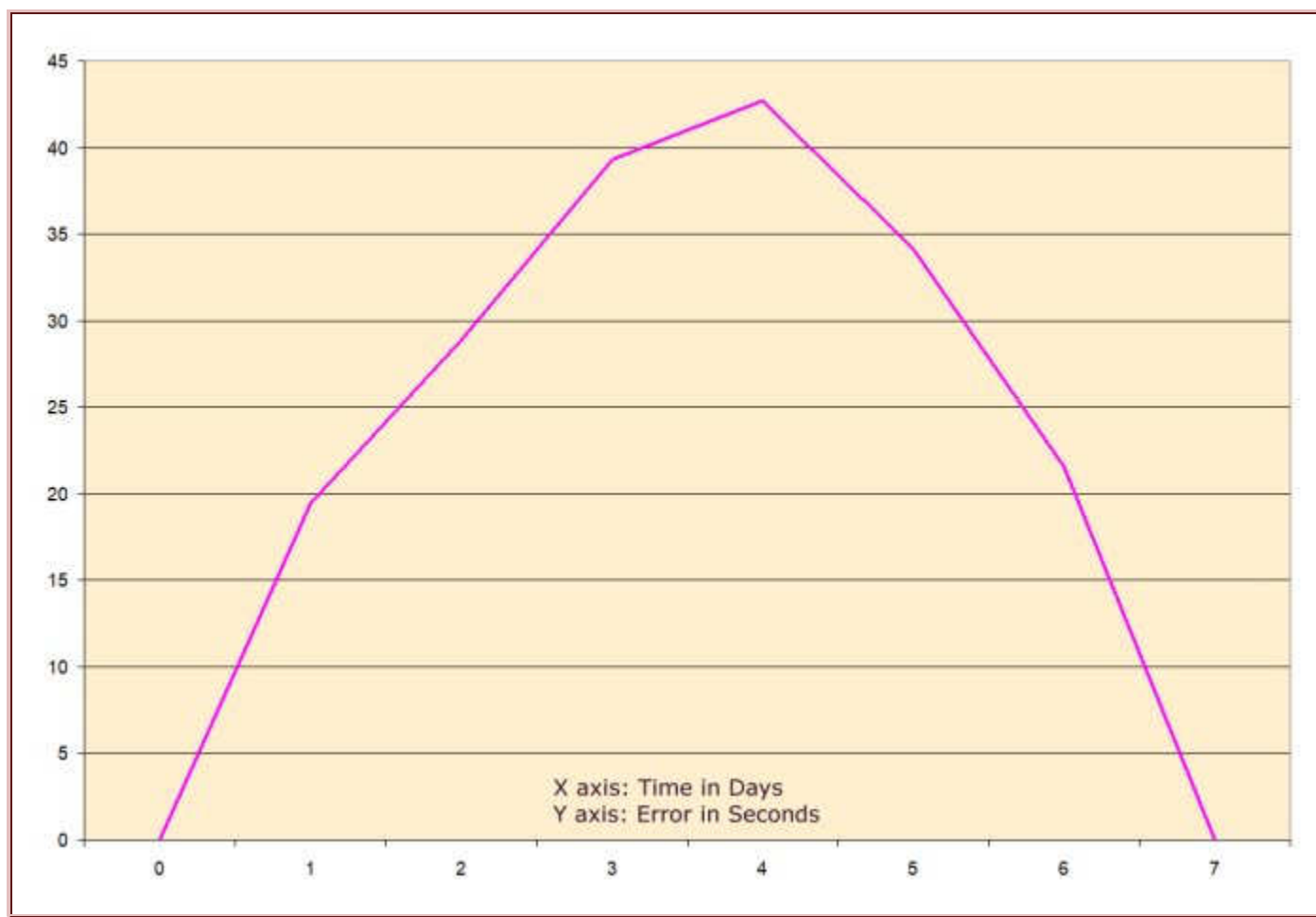
If you adjust the pendulum such that the clock keeps accurate time for the first four days, and wind the mainspring twice a week, you will notice a big improvement in overall accuracy. In this example, the peak error goes down from 43 to 9 seconds, a decrease of 79 percent.
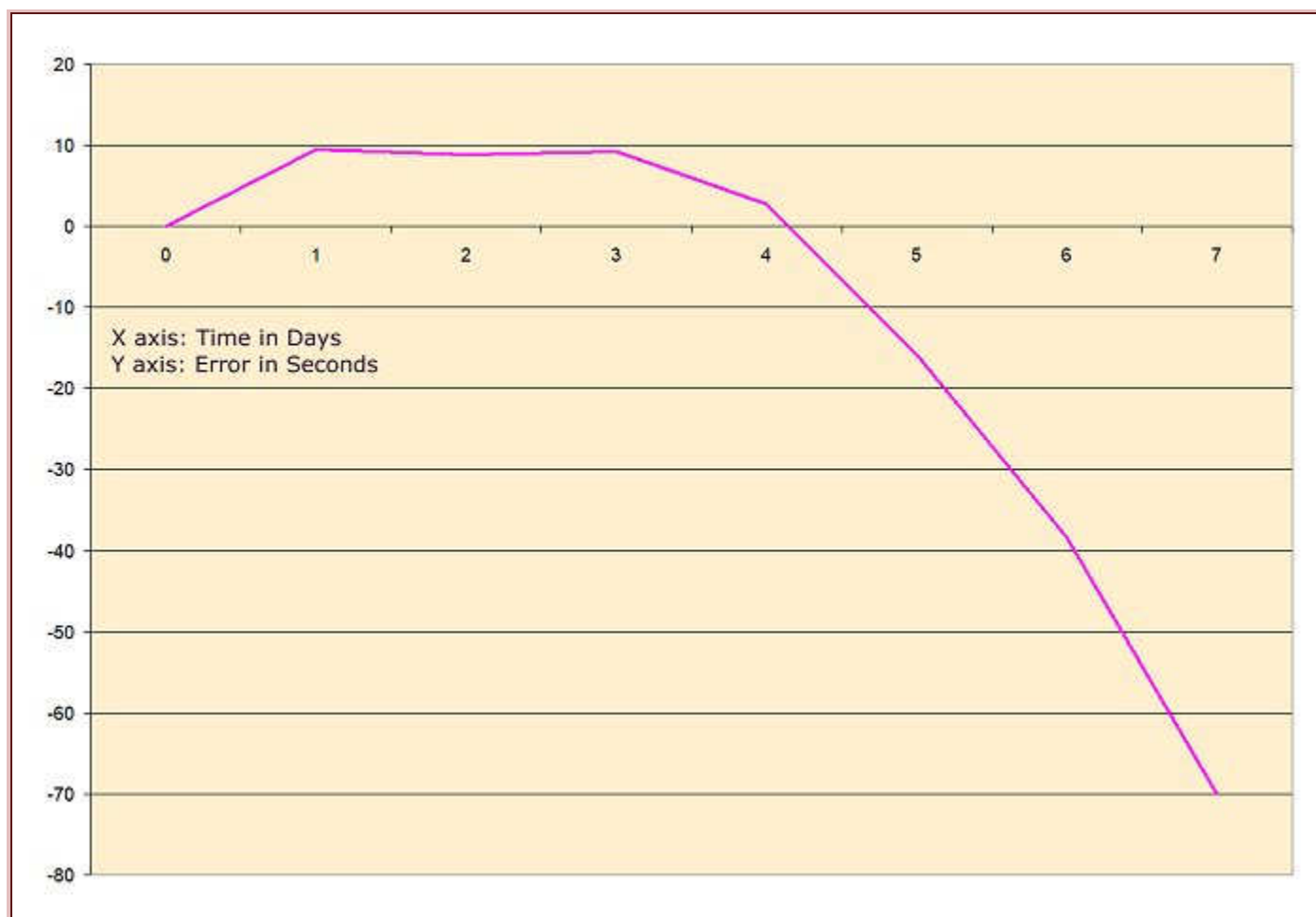
I would like to thank Dr. Bernstein for his extraordinary patience, guidance and support.

Dennis S. Bernstein
ScB, MSE, PhD
Computer, Information and Control Engineering
James E. Knott Professor
Department of Aerospace Engineering
University of Michigan Ann Arbor

Mark Headrick